

# Calculating symmetries in Newman-Tamburino metrics

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**Abstract:** In this paper I show that the Newman-Tamburino spherical metrics always admit a Killing vector, correcting a claim by Collinson and French, (1967 *J. Math. Phys.* **8** 701) and also admit a homothety. A similar calculation is given for the limit of the Newman-Tamburino cylindrical metric.

## 1 Introduction

The Newman-Tamburino metrics are those vacuum solutions of the Einstein equations admitting hypersurface orthogonal geodesic rays with non-vanishing shear and divergence. In the Newman-Penrose formalism this implies that  $\Psi_0 = \kappa = 0$ , that  $\rho$  is real and non-zero and  $\sigma \neq 0$ . In [1] Newman and Tamburino explicitly gave all such metrics and showed that they fall into two classes: the spherical, with  $\rho^2 \neq \sigma\bar{\sigma}$  and the cylindrical with  $\rho^2 = \sigma\bar{\sigma}$ . In [2] Collinson and French claimed to have shown that the former metrics admit at most one Killing vector, and that happens only in a particular subcase. In fact, the spherical Newman-Tamburino metrics always admit a Killing vector and also always admit a homothety. This preprint is intended to show the full calculations and results when the homothetic equations of [3] are integrated for the Newman-Tamburino spherical metrics. The bulk of sections 3 and 4 come from Maple 9 worksheets, exported to T<sub>E</sub>X and suitably tidied up for better readability.

Throughout I use the spin coefficient notation of [4]. For example I use  $\kappa'$ ,  $\rho'$ ,  $\sigma'$  and  $\tau'$  in place of the more traditional  $-\nu$ ,  $-\mu$ ,  $-\pi$  and  $-\lambda$ .

## 2 Results

The contravariant form of the Newman-Tamburino spherical metric [1] (see also [5], equation (26.21)) is

$$\begin{aligned} g^{22} &= -\frac{2r^2(\zeta\bar{\zeta})^{1/2}}{R^2} + \frac{2rL}{A} + \frac{2r^3A(\zeta^2 + \bar{\zeta}^2)}{R^4} - \frac{4r^2A^2(\zeta\bar{\zeta})^{3/2}}{R^4} \\ g^{23} &= 4A^2(\zeta\bar{\zeta})^{3/2}x \left[ \frac{L}{2a^3} - \frac{r-2a}{2a^2R^2} - \frac{r-a}{R^4} \right] \\ g^{24} &= 4A^2(\zeta\bar{\zeta})^{3/2}y \left[ \frac{L}{2a^3} - \frac{r+2a}{2a^2R^2} - \frac{r+a}{R^4} \right] \\ g^{33} &= -\frac{2(\zeta\bar{\zeta})^{3/2}}{(r+a)^2} \quad g^{44} = -\frac{2(\zeta\bar{\zeta})^{3/2}}{(r-a)^2} \quad g^{12} = 1. \end{aligned}$$

Here our coordinates are  $x^1 = u$ ,  $x^2 = r$ ,  $x^3 + ix^4 = x + iy = \zeta$  and

$$A(u) = bu + c, \quad L = \frac{1}{2} \log \left( \frac{r+a}{r-a} \right) \quad a = A(\zeta \bar{\zeta})^{1/2} \quad R^2 = r^2 - a^2.$$

Here  $b$  and  $c$  are real constants.

The Collinson and French result (also quoted in [5]) is that there is a Killing vector only in the case where  $A$  is constant — in this situation the Killing vector is the obvious  $\partial_u$ . However, if  $b \neq 0$  we can set  $c = 0$  by a coordinate change and then the vector

$$K^a = -u\partial_u + r\partial_r + 2x\partial_x + 2y\partial_y.$$

is a Killing vector, as will be shown in section 3. This can be checked directly: consider the flow of  $K^a$ . This scales the coordinates by

$$u \rightarrow \lambda^{-1}u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda^2 \zeta$$

for real parameter  $\lambda > 0$ . Under this scaling it is easy to check that all the contravariant components given above are homogeneous in  $\lambda$  (when  $A = bu$ ), and all of the correct degree to make the flow isometric. For example, the  $g^{22}$  component is homogeneous of degree 2, and so the metric term  $g^{22} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r}$  is unchanged under the flow.

Also the vector

$$H = r\partial_r + x\partial_x + y\partial_y$$

is a homothety, whatever  $A$  is (see section 3). Alternatively, the flow of  $H$  is

$$u \rightarrow u, \quad r \rightarrow \lambda r, \quad \zeta \rightarrow \lambda \zeta,$$

and we again find that all the contravariant components given above are homogeneous in  $\lambda$ , and all of the correct degree to make the flow homothetic. For example, the  $g^{22}$  component is homogeneous of degree 1, and so the metric term  $g^{22} \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r}$  scales by  $\lambda^{-1}$ : the same scaling applies to all the metric terms.

Newman and Tamburino [1] also give the following metric, which arises as a limit of the cylindrical case (see also [5] (26.23) for corrections to the  $du^2$  coefficient):

$$ds^2 = 2 du dr - x^{-2} [b + \log(r^2 x^4)] du^2 + 4 \frac{r}{x} du dx - r^2 dx^2 - x^2 dy^2,$$

with the same coordinates as used in the spherical case. The Killing vectors here are obvious ( $\partial_u$  and  $\partial_y$ ) and as we shall see there is also a homothetic vector (see section 4)

$$H_2 = 2r\partial_r - x\partial_x + 2y\partial_y.$$

One can use the flow of  $H_2$  to check it is a homothety as well.

### 3 The Calculations (spherical case)

The basic information is taken from Collinson and French [2], and Newman and Tamburino [1]. See those papers for those spin coefficients that are not actually calculated here. I have checked in a separate calculation that their results are correct as quoted. I use as coordinates  $u, r, \zeta = x + iy$ .

Collinson and French [2] wrote the conformal Killing equations in Newman-Penrose form and used that in their work, although there are a few minor typos in their paper. Here, I will use the formalism of [3], which generalised the ideas of [6] into a form suitable for this task. I will use to the notation of [3] for the components of the homothety

$$\xi_a = \xi_n \ell_a + \xi_\ell n_a - \xi_{\bar{m}} m_a - \xi_m \bar{m}_a,$$

and its bivector,  $F_{ab}$ , with anti-self dual

$${}^-F_{ab} = 2\phi_{00} \ell_{[a} m_{b]} + 2\phi_{01} (\ell_{[a} n_{b]} - m_{[a} \bar{m}_{b]}) - 2\phi_{11} n_{[a} \bar{m}_{b]}.$$

The tetrad is a standard tetrad (see [1]), based around the Debever-Penrose vector  $\ell^a = \partial_r$ , see [1] and [2] for further detail. Since the tetrad is normalised, for the Penrose-Rindler spin coefficients used in [3] we have  $\gamma' = -\epsilon$ ,  $\beta' = -\alpha$  etc.

In the Maple I use use **z** for  $\zeta$  and **w** for  $\bar{\zeta}$ ; **H1** for  $\xi_\ell$  etc. I typically add a **b** for a complex conjugate (**Hmb** is  $\xi_{\bar{m}}$ ) and a **1** for a dash (**rho1** is  $\rho'$ ).

Firstly, define the terms **a**, **a0** (the latter is  $\alpha_0$  in [2]).

```
> a:=A(u)*z^(1/2)*w^(1/2):
> a0:=3/4*w^(3/4)*z^(-1/4):
> a0b:=3/4*z^(3/4)*w^(-1/4):
```

Rather than use the explicit definitions for  $L$  and  $R$  in [2] and [1], I will leave them as “unknown” functions and define a routine later that will substitute for their derivatives. I will also use  $Q(u, r, z, w)$  in place of  $1/R^2$  to make things more transparent. I define what these functions actually are so we can substitute for them more easily when that become useful. I also define dummy symbols to use in place of the full functional dependence of  $L$  and  $Q$  for ease of readability. I have also suppressed the functional dependence in the Maple output, replacing  $Q(u, r, z, w)$  with  $Q(x^a)$  for example.

```
> LL:=L(u,r,z,w):Lis:=1/2*log((r+a)/(r-a));
```

$$Lis := \frac{1}{2} \ln \left( \frac{r + A(u) \sqrt{z} \sqrt{w}}{r - A(u) \sqrt{z} \sqrt{w}} \right)$$

```
> QQ:=Q(u,r,z,w):Qis:=1/(r^2-a^2);
```

$$Qis := (r^2 - (A(u))^2 zw)^{-1}$$

Now we define the routine to simplify derivatives and products and also add a line to collect terms.

```

> diffsubs:=proc(XX)
> subs(diff(L(u,r,z,w),r)=-a*Q(u,r,z,w),XX):
> subs(diff(L(u,r,z,w),u)=r*Q(u,r,z,w)*diff(a,u),%);
> subs(diff(L(u,r,z,w),w)=r*Q(u,r,z,w)*diff(a,w),%):
> subs(diff(L(u,r,z,w),z)=r*Q(u,r,z,w)*diff(a,z),%):
> subs(diff(Q(u,r,z,w),r)=-2*r*Q(u,r,z,w)^2,%):
> subs(diff(Q(u,r,z,w),z)=2*Q(u,r,z,w)^2*a*diff(a,z),%):
> subs(diff(Q(u,r,z,w),u)=2*Q(u,r,z,w)^2*a*diff(a,u),%):
> subs(diff(Q(u,r,z,w),w)=2*Q(u,r,z,w)^2*a*diff(a,w),%):
> student[powsubs](r^2=a^2+1/Q(u,r,z,w),expand(%));
> collect(%, [L(u,r,z,w),Q(u,r,z,w),r,Hl(u),psi,z,w]);
> end proc:

```

The terms  $S$  and  $Sb$  are  $\psi_0^1$  and its conjugate in [2].

```

> S:=2*A(u)^2*z^(3/4)*w^(3/4)*z; Sb:=2*A(u)^2*z^(3/4)*w^(3/4)*w:

```

$$S := 2 A(u)^2 z^{7/4} w^{3/4}$$

And the curvature component  $\Psi_1$  is given in [1].

```

> Psi1:=S*QQ^2;

```

$$\Psi_1 := 2 A(u)^2 z^{7/4} w^{3/4} Q(x^a)^2$$

Now from [2] we have  $\kappa = \epsilon = \tau' = \Psi_0 = 0$ , and  $\rho$  and  $\sigma$  real — these can also be easily checked by Maple. So by [3](6a)  $D\xi_\ell = 0$ . Using  $\tau = \bar{\alpha} + \beta$  ([2]), [3](6c) becomes

$$\delta\xi_\ell = \tau\xi_\ell - \rho\xi_m - \sigma\xi_{\bar{m}} - \phi_{11}.$$

We next use equation [3](11), since  $\ell^a$  is a Debever-Penrose vector. Unfortunately, [3](11) contains an error — the right hand side is the complex conjugate of what it ought to be. With this correction, we have

$$\phi_{11} = -\tau\xi_\ell + \rho\xi_m + \sigma\xi_{\bar{m}}.$$

Hence  $\delta\xi_\ell = 0$  and  $\xi_\ell = \xi_\ell(u)$ , as found in [2].

Equation (10a) of [3] is

$$D\phi_{11} = -\xi_\ell\Psi_1,$$

and so integrates to give the  $r$  dependence of  $\phi_{11}$ , here called  $p_{11}$ . We ignore the factor independent of  $r$  when integrating:

```

> int(-Qis^2,r);

```

$$-\frac{r}{2A(u)^2zw(r^2 - A(u)^2zw)} + \frac{1}{2}\operatorname{arctanh}\left(\frac{r}{A(u)\sqrt{zw}}\right)A(u)^{-3}z^{-1}w^{-1}\frac{1}{\sqrt{zw}}$$

The  $\operatorname{arctanh}$  term here is just  $L$  and we get

```

> p11:=S*Hl(u)/2/a^2*r*QQ-S*Hl(u)/2/a^3*LL+p110(u,z,w);

```

$$p_{11} := \frac{z^{3/4}Hl(u)rQ(x^a)}{w^{1/4}} - \frac{z^{1/4}Hl(u)L(x^a)}{A(u)w^{3/4}} + p_{110}(u,z,w)$$

Here  $p110(u, z, w)$  is the integration constant. Now the spin coefficients — see [2].

```
> alpha:=expand(simplify(S*LL*r/2/a^2*QQ+QQ*(r*a0+a*a0b)-S*QQ/a/2,radical)):
> beta:=-S*LL*QQ/2/a-QQ*(r*a0b+a*a0):
Now a routine to take conjugates nicely, as we need conjugates to define  $\tau$ .
> conj:=proc(XX)
> subs(z=w1,w=z1,XX):
> subs(w1=w,z1=z,%):subs(I=-I,%);
> subs(L(u,r,w,z)=L(u,r,z,w),Q(u,r,w,z)=Q(u,r,z,w),%);
> subs(p110(u,w,z)=p110b(u,z,w),p110b(u,w,z)=p110(u,z,w),%);
> end proc:
> conj(alpha)+beta:
> tau:=collect(%,[L(u,r,z,w),Q(u,r,z,w)]);

$$\tau := (-A(u) z^{5/4} w^{1/4} + r w^{3/4} z^{-1/4}) Q(x^a) L(x^a) - A(u) w^{5/4} z^{1/4} Q(x^a)$$

> rho:=-r*Q(u,r,z,w):
> sigma:=a*Q(u,r,z,w):
> sigma1:=r*LL^2*S^2/4/a^4*QQ+r*LL*S*Sb/2/a^4*QQ-r*diff(a,u)*QQ-S*Sb/2/a^3*QQ:
> unprotect(gamma);
> gamma:=-r*LL^2*S*Sb/4/a^4*QQ+S^2*LL^2/4/a^3*QQ+a*r*LL*(S*a0b-Sb*a0)/2/a^3*QQ
> + S*Sb*QQ*LL/2/a^3 - S*Sb/4/a^3*(LL/2/a^2-r/2/a*QQ) +
> (S^2/2/a-a*(S*a0b-Sb*a0))/2/a^2*QQ:
```

We need derivative operators  $\delta$  and  $\delta'$  to find  $\rho'$ . Firstly, the components of  $m^a$  come from [2] and [1].

```
> om0:=-A(u)*z^(1/4)*w^(5/4):
> omega:=-Sb*LL/2/a^2+(r*om0-a*conj(om0))*QQ;

$$\omega := -w^{3/4} z^{-1/4} L(x^a) + (-r A(u) z^{1/4} w^{5/4} + A(u)^2 z^{7/4} w^{3/4}) Q(x^a)$$

> omega1:=conj(omega):
> P:=z^(3/4)*w^(3/4):
> del:=XX->omega*diff(XX,r)+2*r*P*QQ*diff(XX,w)-2*a*P*QQ*diff(XX,z):
> del1:=XX->omega1*diff(XX,r)+2*r*P*QQ*diff(XX,z)-2*a*P*QQ*diff(XX,w):
To calculate  $\rho'$  we use [4] (4.11.12  $e'$ ).
> expand((diff(sigma1,r)-rho*sigma1)/sigma):
> rho1:=diffsbs(%);
```

$$\rho 1 := -A(u) z^2 L(x^a)^2 Q(x^a) - 2 \left( A(u) z w + \frac{z^{3/2} r}{\sqrt{w}} \right) Q(x^a) L(x^a) + A(u) z w \frac{dA}{du} Q(x^a)$$

We check some curvature equations next before we go on.

```
> diffsbs(diff(tau,r) - rho*tau-sigma*conj(tau)-Psi1); #[4]4.11.12c
```

0

```

> diffsubs(diff(alpha,r)-rho*alpha-sigma*beta); # [4] 4.11.12h & i';
0
> diffsubs(diff(beta,r)-sigma*alpha-beta*rho-Psi1); # [4] 4.11.12h' & i;
0
> Psi2:=-diffsubs(diff(rho1,r)-rho1*rho-sigma*sgma1) ; # [4] 4.11.12 f'

$$\Psi_2 := -4A(u)^2 z^{5/2} L(x^a) Q(x^a)^2 \sqrt{w} - (2A(u) z^2 r + 4A(u)^2 z^{3/2} w^{3/2}) Q(x^a)^2$$

This expression for  $\Psi_2$  agrees with [1].
> diffsubs(diff(gamma,r) -beta*conj(tau)-alpha*tau-Psi2); # [4] 4.11.12k
0
> diffsubs(del(rho)-del1(sigma)-rho*(conj(alpha)+beta)+
> sigma*(3*alpha-conj(beta) ) + Psi1); # [4] 4.11.12 d
0
> diffsubs( del1(beta)-del(alpha)-rho*rho1+sigma*sigma1+alpha*conj(alpha)+
> beta*conj(beta)-2*alpha*beta-Psi2 ); # [4] 4.11.12 l
0

```

Integrating [3] (6g) and using [3](11) (corrected, see above):

```

> Hm1:=-r*p110(u,z,w) + Hl(u)*S/2/a^3*r*LL+Hm0;

```

$$Hm1 := -rp110(u, z, w) + \frac{Hl(u) z^{1/4} r L(x^a)}{A(u) w^{3/4}} + Hm0$$

```

> Hmb1:=-r*p110b(u,z,w) + Hl(u)*Sb/2/a^3*r*LL+Hmb0;

```

$$Hmb1 := -rp110b(u, z, w) + \frac{Hl(u) w^{1/4} r L(x^a)}{A(u) z^{3/4}} + Hmb0$$

By [3] (11) the following ought to be zero.

```

> diffsubs(p11+tau*Hl(u)-rho*Hm1-sigma*Hmb1):
> collect(%/QQ,r);

```

$$\left( Hm0 + \frac{z^{3/4} Hl(u)}{w^{1/4}} + A(u) \sqrt{z} \sqrt{w} p110b(u, z, w) \right) r - A(u) \sqrt{z} \sqrt{w} Hmb0 - (A(u))^2 zw p110(u, z, w) - Hl(u) A(u) w^{5/4} z^{1/4}$$

```

> expand(solve(coeff(%,r,1),Hm0)),expand(solve(coeff(%,r,0),Hmb0));

```

$$-\frac{z^{3/4} Hl(u)}{w^{1/4}} - A(u) \sqrt{z} \sqrt{w} p110b(u, z, w), -\sqrt{z} \sqrt{w} A(u) p110(u, z, w) - \frac{w^{3/4} Hl(u)}{z^{1/4}}$$

So we get

```

> Hm:=-r*p110(u,z,w)-a*p110b(u,z,w)+Hl(u)*expand(S/2/a^3*(r*LL-a)):

```

> Hmb:=conj(Hm):

These agree with the components in [2] (their  $V_3$  and  $V_4$ ). Now we use [3] (10b) to get  $\phi_{01}$ .

> diffsubs(Psi1\*Hm/2/sigma-beta\*p11/sigma+del(p11)/2/sigma):

> p01:=collect(%,[Hl(u),Q(u,r,z,w),L(u,r,z,w),r]);

$$p01 := \left( \left( \frac{1}{2} \sqrt{z} \sqrt{w} r + \left( 2 \frac{z^{3/2} r}{\sqrt{w}} + A(u) z w \right) L(x^a) \right) Q(x^a) - \frac{z (L(x^a))^2}{A(u) w} - \frac{L(x^a)}{2A(u)} \right) Hl(u) \\ + \frac{z^{3/4} L(x^a) p110(u, z, w)}{w^{1/4}} + \frac{3w^{3/4} p110(u, z, w)}{4z^{1/4}} \\ - \left( A(u) z^{5/4} w^{1/4} r p110(u, z, w) + A(u)^2 z^{7/4} w^{3/4} p110b(u, z, w) \right) Q(x^a) \\ + \left( \frac{3p110(u, z, w)}{4A(u) w^{3/4}} + \frac{w^{1/4} \frac{\partial}{\partial w} p110(u, z, w)}{A(u)} \right) z^{1/4} r - z^{3/4} w^{3/4} \frac{\partial}{\partial z} p110(u, z, w)$$

Now [3](10c) and (8a) will give us information on the  $w$  (that is,  $\bar{\zeta}$ ) dependence of  $\phi_{11}^0$ .

> diffsubs(Psi1\*Hmb-Psi2\*Hl(u)+2\*rho\*p01+2\*alpha\*p11-del1(p11)); # [3] 10c

$$\left( \frac{3p110(u, z, w)}{2A(u) w^{3/4}} - 2 \frac{w^{1/4}}{A(u)} \frac{\partial}{\partial w} p110(u, z, w) \right) z^{1/4}$$

> dsolve(%=0,p110(u,z,w));

$$p110(u, z, w) = \frac{-F1(u, z)}{w^{3/4}}$$

> diffsubs(diff(p01,r)+del1(p11)-2\*rho\*p01-2\*alpha\*p11); # [3] (8a)

$$\left( \frac{9p110(u, z, w)}{4A(u) w^{3/4}} + 3 \frac{w^{1/4}}{A(u)} \frac{\partial}{\partial w} p110(u, z, w) \right) z^{1/4}$$

> dsolve(%=0,p110(u,z,w));

$$p110(u, z, w) = \frac{-F1(u, z)}{w^{3/4}}$$

Both giving the same result. Now we turn to  $\xi_n$  and [3](6i), which we solve for  $\sigma \xi_\ell$ .

> rhs6i:=diffsubs((-del(Hm)-conj(sigma1)\*Hl(u)-Hm\*(conj(alpha)-beta))):

The imaginary part ought to be zero as  $\xi_n$  is real, so using results from [3] (10c) and (8a), we find the imaginary part divide out a common non-zero factor and call what's left  $X$ .

```

> Imrhs6i:=%-conj(%):
> subs(p110(u,z,w)=F(u,z)/w^(3/4),p110b(u,z,w)=Fb(u,w)/z^(3/4),%):
> X:=expand(%/r/qq/sqrt(z)/sqrt(w));

```

$$X := -4 z^{3/4} A(u) \frac{\partial}{\partial z} F(u, z) + 4 w^{3/4} A(u) \frac{\partial}{\partial w} Fb(u, w) + 3 \frac{A(u) Fb(u, w)}{w^{1/4}} - 3 \frac{A(u) F(u, z)}{z^{1/4}}$$

Assuming  $F$  is differentiable in  $z$  we can split this

```

> subs(Fb=0,X):%;

```

$$-4 z^{3/4} A(u) \frac{\partial}{\partial z} F(u, z) - 3 \frac{A(u) F(u, z)}{z^{1/4}}$$

This is a (real) function of  $u$  and  $w$ . We choose the shape of the separation function to simplify the solution to the differential equation slightly.

```

> dsolve(=-4*A(u)*G(u),F(u,z));

```

$$F(u, z) = \frac{G(u)z + F1(u)}{z^{3/4}}$$

Check this out:

```

> subs(F(u,z)=G(u)*z^(1/4)+H(u)/z^(3/4),Fb(u,w)=G(u)*w^(1/4)+Hb(u)/w^(3/4),X):
> expand(%);

```

0

So we define a simplification routine for  $\phi_{11}^0$ .

```

> P110sbs1:=proc(XX)
> subs(p110(u,z,w)=F(u,z)/w^(3/4),p110b(u,z,w)=Fb(u,w)/z^(3/4),XX);
> subs(F(u,z)=G(u)*z^(1/4)+H(u)/z^(3/4),Fb(u,w)=G(u)*w^(1/4)+Hb(u)/w^(3/4),%);
> expand(%);
> end proc;

```

And check it works

```

> P110sbs1(Imrhs6i);

```

0

Turning to [3](6b) next,

```

> eqn6b:=H1(u)*(gamma+conj(gamma))-conj(tau)*Hm-tau*conj(Hm)-p01-conj(p01)+psi:
> P110sbs1(diffsbs(%));

```

$$\psi - G(u) - 3 \frac{H(u)}{2z} - 3 \frac{Hb(u)}{2w}$$

This ought to be  $\dot{\xi}_\ell$ , a function of  $u$  only, so  $H = 0$  and we define a new simplification routine and test it out:

```

> P110sbs2:=proc(XX);
> expand( subs( p110(u,z,w)= (psi-diff(H1(u),u))*z^(1/4)/w^(3/4) ,
> p110b(u,z,w)=(psi-diff(H1(u),u))*w^(1/4)/z^(3/4),XX));
> collect(%,[L(u,r,z,w),Q(u,r,z,w),r,H1(u),z,w]);end proc;
> P110sbs2(diffsbs(eqn6b));

```



$$\frac{d}{du} Hl(u)$$

This is as it should be. Now we can define  $\xi_n$ .

```
> P110sbs2(diffsbs(rhs6i+conj(rhs6i))/sigma/2):
> Hn:=collect(%,[L(u,r,z,w),Q(u,r,z,w),Hl(u),diff(Hl(u),u),r,psi]):
```

We check this against the [2] version, called  $V_2$  there. It is clear from the shape of  $\xi_m$  ( $= V_3$  of [2]) that  $a_0$  in [2] is my  $\phi_{11}^0$ .

```
> ay0:=p110(u,z,w):ay0b:=p110b(u,z,w):
> V2:=r*LL^2*(-S^2-Sb^2)*Hl(u)/4/a^5 - LL^2*S*Sb*Hl(u)/4/a^4 -
> LL*(ay0b*S+ay0*Sb)/2/a + r*(2*a*Hl(u)*diff(a,u)-ay0*Sb-ay0b*S)/2/a^2 +
> (-2*a^3*ay0*S-2*a^3*ay0b*Sb+Hl(u)*S*Sb)/4/a^4+r*LL*(-2*a^3*ay0*S-
> 2*a^3*ay0b*Sb- Hl(u)*S*Sb)/4/a^5 + 1/QQ*(-3*ay0*S-3*ay0b*Sb-
> 4*a^2*2*P*(diff(p110(u,z,w),w) +diff(p110b(u,z,w),z) ) )/8/a^3:
> expand(Hn-P110sbs2(V2)):
> simplify(subs(psi=0,diff(Hl(u),u)=0,%));
```

0

So our  $\xi_n$  agrees with [2] in the case of their Killing vector ( $\psi = 0$  and  $\xi_\ell$  constant). However, if  $\xi_\ell$  is not constant, the terms differ:

```
> simplify(subs(psi=0,%));
```

$$-\frac{2}{z^{3/2}w^{3/2}} \frac{dHl(u)}{du} \left( (w^3z + z^3w + 2z^2w^2L(x^a)) A(u) + L(x^a) r(z^{5/2}\sqrt{w} + w^{5/2}\sqrt{z}) + 2rz^{3/2}w^{3/2} \right)$$

Next, we put our  $\xi_n$  into [3](6d).

```
> eqn6d:=diffsbs(diff(Hn,r)-p01-conj(p01)-psi):
> P110sbs2(%);
```

$$\frac{Hl(u) \frac{d}{du} A(u)}{A(u)} - \frac{d}{du} Hl(u)$$

```
> dsolve(%,Hl(u));
```

$$Hl(u) = \_C1 A(u)$$

So next a routine to replace  $\xi_\ell(u)$  with a multiple of  $A(u)$ , and also to kill off the second derivative of  $A(u)$ .

```
> Hlsbs:=proc(XX)
> subs(Hl(u)=C*A(u),XX);subs(diff(A(u),u,u)=0,%);subs(diff(Hl(u),u,u)=0,%);%;
> end proc:
```

We now try [3](6j).

```
> del1(Hm)+conj(rho1)*Hl(u)+rho*Hn+(conj(beta)-alpha)*Hm-p01+conj(p01)+psi:
> Hlsbs(P110sbs2(diffsbs(%)));
```

0

So that is satisfied. Now for  $\phi_{00}$ , which we get from the conjugate of [3](6f).

```
> eqn6f:=-del1(Hn)-(conj(beta)+alpha)*Hn-conj(rho1)*Hmb-sigma1*Hm:
> p00:=diffsbs(P110sbs2(diffsbs(%))):
```

I've suppressed this component as it's very long, but we check the result with [3] (8d).

```
> del1(p01)+diff(p00,r)-rho*p00-sigma1*p11:
> P110sbs2(diffsbs(%));
```

0

To go any further we need to get the components of  $n^a = (1, U, X^3, X^4)$  and to define  $D'$ . Taking the metric terms from [1] and [2]:

```
> gup22:=-2*r^2*sqrt(w)*sqrt(z)*QQ + 2*r*LL/A(u) + QQ^2*(2*r^3*A(u)*(w^2+z^2)
- 4*r^2*A(u)^2*w^(3/2)*z^(3/2)):
> gup22+2*omega*conj(omega):
> U:=diffsbs(%/2);
```

$$U := \sqrt{z}\sqrt{w}L(x^a)^2 + ((w^2 + z^2)A(u)r - 2A(u)^2z^{3/2}w^{3/2})Q(x^a) - \sqrt{z}\sqrt{w} \\ + \left( (-A(u)^2z^{5/2}\sqrt{w} + 2zwrA(u) - A(u)^2\sqrt{z}w^{5/2})Q(x^a) + \frac{r}{A(u)} \right)L(x^a)$$

```
> gup33:=-2*z^(3/2)*w^(3/2)/(r+a)^2:gup44:=-2*z^(3/2)*w^(3/2)/(r-a)^2:
```

These next two terms are the components of  $m^a$ .

```
> xi3:=P*(r-a)*QQ;xi4:=I*P*(r+a)*QQ;
```

$$xi3 := z^{3/4}w^{3/4}(r - A(u)\sqrt{z}\sqrt{w})Q(x^a) \\ xi4 := iz^{3/4}w^{3/4}(r + A(u)\sqrt{z}\sqrt{w})Q(x^a)$$

```
> xi3*conj(xi4)+xi4*conj(xi3); # checking
```

0

```
> simplify((subs(Q(u,r,z,w)=Qis,xi3*conj(xi3)*2+gup33)));
```

0

```
> simplify(subs(Q(u,r,z,w)=Qis,xi4*conj(xi4)*2+gup44));
```

0

```
> gup23:=4*A(u)^2*z^(3/2)*w^(3/2)*(z+w)/2*(LL/2/a^3-(r-2*a)*QQ/2/a^2
> -(r-a)*QQ^2):
```

```
> gup24:=4*A(u)^2*z^(3/2)*w^(3/2)*(z-w)/2/I*(LL/2/a^3-(r+2*a)*QQ/2/a^2
> -(r+a)*QQ^2):
```

```
> gup23+omega*conj(xi3)+conj(omega)*xi3:
```

```
> X3:=diffsbs(%);
```

$$X3 := \left( \left( (-z^{3/2}\sqrt{w} - \sqrt{z}w^{3/2})r + A(u)zw^2 + A(u)z^2w \right) Q(x^a) + \frac{w}{A(u)} + \frac{z}{A(u)} \right) L(x^a) \\ + \left( (-z^{3/2}\sqrt{w} - \sqrt{z}w^{3/2})r + A(u)zw^2 + A(u)z^2w \right) Q(x^a)$$

```
> gup24+omega*conj(xi4)+omega1*xi4:
> X4:=diffsbs(factor(%));
```

$$X4 := i \left( \left( (-z^{3/2}\sqrt{w} + \sqrt{z}w^{3/2})r + zw^2A(u) - z^2wA(u) \right) Q(x^a) + \frac{w}{A(u)} - \frac{z}{A(u)} \right) L(x^a) \\ + \left( (-i\sqrt{z}w^{3/2} + iz^{3/2}\sqrt{w})r + iz^2wA(u) - izw^2A(u) \right) Q(x^a)$$

As a double check we firstly define the (contravariant) tetrad and then check against the metric terms.

```
> ell:=<0,1,0,0>;en:=<1,U,X3,X4>;
> em:=<0,omega,xi3,xi4>;emb:=map(conj,em):
> ell.Transpose(en)-em.Transpose(emb):
> %+Transpose(%):
> g:=map(diffsbs,%):
> diffsbs(g[2,2]-gup22);
0
> diffsbs(g[2,3]-gup23);
0
> diffsbs(g[2,4]-gup24);
0
> diffsbs(simplify(g[3,3]-gup33));
0
> diffsbs(simplify(g[4,4]-gup44));
0
```

For a second check we apply two of the commutators [4] (4.11.11) to  $r$  and check what we get.

```
> diff(U,r)+gamma+conj(gamma)-tau*conj(omega)-conj(tau)*omega:
> diffsbs(%);
0
> diff(X3,r)-tau*conj(xi3)-conj(tau)*xi3:
> diffsbs(%);
0
```

Since all this checks out we go ahead and define  $D'$ .

```
> D1:=proc(XX)
> diff(XX,u)+diff(XX,r)*U+(X3+I*X4)*diff(XX,z)+(X3-I*X4)*diff(XX,w);
> P110sbs2(diffsbs(%));
> end proc;
```

We make use of  $D'$  firstly to find the last spin coefficient,  $\kappa'$ , using [4](4.11.12g).

```
> D1(beta)-del(gamma)-tau*rho1-alpha*conj(sigma1)-beta*(rho1+gamma-conj(gamma))
> +gamma*(tau-beta-conj(alpha)): # should be -kappa'*sigma
> diffsb2(%):
> kappa1:=-diffsb2(P110sbs2(%/sigma));
```

$$\begin{aligned} \kappa_1 := & - \left( z^{5/4} w^{1/4} r - \frac{z^{11/4} A(u)}{w^{1/4}} \right) Q(x^a) L(x^a)^3 \\ & - \left( \left( -2 \frac{z^{9/4}}{w^{3/4}} + 2 z^{1/4} w^{5/4} \right) r - 3 z^{7/4} w^{3/4} A(u) \right) Q(x^a) L(x^a)^2 + \\ & \left( \left( \frac{dA}{du} z^{1/4} w^{5/4} - 2 z^{5/4} w^{1/4} \right) r - A(u) \left( z^{7/4} w^{3/4} \frac{dA}{du} + 4 z^{3/4} w^{7/4} \right) \right) Q(x^a) L(x^a) \\ & - \left( -2 z^{7/4} w^{3/4} A(u) + w^{7/4} z^{3/4} A(u) \frac{dA}{du} - 2 z^{5/4} w^{1/4} r \frac{dA}{du} \right) Q(x^a) \end{aligned}$$

Now to look at the equations that involve  $D'$ . Firstly [3] (6h):

```
> eqn6h:=D1(Hm)+conj(kappa1)*H1(u)+tau*Hn+(conj(gamma)-gamma)*Hm-conj(p00):
> P110sbs2(diffsbs(%)):factor(Hlsbs(%));
```

0

Then we look at [3] (6e):

```
> eqn6e:=D1(Hn)+(gamma+conj(gamma))*Hn+kappa1*Hm+conj(kappa1)*Hmb:
> P110sbs2(diffsbs(%)):
> factor(Hlsbs(%));
```

0

and [3] (8c)

```
> eq8c:=diffsb2(del(p01)+D1(p11)-sigma*p00-2*tau*p01-(rho1+2*gamma)*p11):
> factor(Hlsbs(P110sbs2(%)));
```

0

and [3] (8b)

```
> D1(p01)+del(p00)-(tau-2*beta)*p00-2*rho1*p01-kappa1*p11:
> P110sbs2(diffsbs(%)):
> factor(Hlsbs(%));
```

0

and finally [3] (10d).

```
> eqn10d:=Psi2*Hm-Psi1*Hn-2*tau*p01-2*gamma*p11+D1(p11):
> P110sbs2(diffsbs(%)):factor(Hlsbs(%));
0
```

Next we consider what happens if we have a Killing vector ( $\psi = 0$ ) with  $\xi_\ell$  zero ( $C = 0$ )

```
> subs(C=0,psi=0,Hlsbs(P110sbs2(Hm)));
0
> subs(C=0,psi=0,Hlsbs(P110sbs2(Hn)));
0
```

Hence we cannot have both  $\psi$  and  $C$  zero. This is the Collinson and French result: only one Killing vector at most. We have a look at the homothety.

```
> H1(u)*en+Hn*ell-Hm*emb-conj(Hm)*em:
> map(diffsbs,%):
> map(P110sbs2,%):
> map(Hlsbs,%):
> subs(z=x+I*y,w=x-I*y,K):K:=map(expand,%);
```

$$K := \begin{pmatrix} CA(u) \\ -rC\frac{dA}{du} + 2r\psi \\ 2\psi x - 2C\frac{dA}{du}x \\ 2\psi y - 2C\frac{dA}{du}y \end{pmatrix}$$

So the obvious Killing vector if  $A$  is constant:

```
> KK:=subs(psi=0,C=1/B,diff(A(u),u)=0,A(u)=B,K):%;
```

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The new Killing vector in the other case:

```
> KK2:=subs(psi=0,A(u)=u*B,C=1/B,K):map(simplify,%);
```

$$\begin{pmatrix} u \\ -r \\ -2x \\ -2y \end{pmatrix}$$

And the proper homothety for both cases:

```
> HH:=subs(C=0,psi=1,K);
```

$$\begin{pmatrix} 0 \\ 2r \\ 2x \\ 2y \end{pmatrix}$$

We now turn to the remaining curvature equations and Bianchi identities. To make life easy, we define the weighted derivative operators, [4] section 4.14.

```

> thorn:=X->dffsbs(diff(X,r)):
> thorn1:=proc(X,p,q)
> local i;
> D1(X)-p*gamma*X-q*conj(gamma)*X;
> dffsbs(expand(%));
> end proc:
> edth:=proc(X,p,q)
> local i;
> del(X)-p*beta*X-q*conj(alpha)*X;
> dffsbs(expand(%))
> end proc:
> edth1:=proc(X,p,q)
> local i;
> del1(X)-p*alpha*X-q*conj(beta)*X;
> dffsbs(expand(%))
> end proc:

```

As a check on the calculations, we can run through the curvature equations, [4] (4.12.32), some of which we've used already, some of which will give us  $\Psi_3$  and  $\Psi_4$ . The only ones that do not give zero are  $(b')$  and  $(c')$ , the first of which gives us  $\Psi_4$ :

```

> thorn1(sigma1,-3,1)-edth1(kappa1,-3,-1)-sigma1*(rho1+conj(rho1))
> +kappa1*(conj(tau)):
> Psi4:=Hlsbs(dffsbs(%));

```

$$\begin{aligned}
\Psi_4 := & -8 L(x^a)^3 z^4 Q(x^a)^2 A(u)^2 \\
& + \left( \left( -12 \frac{r z^{7/2} A(u)}{\sqrt{w}} - 32 A(u)^2 z^3 w \right) Q(x^a)^2 - 8 z^2 Q(x^a) \right) L(x^a)^2 \\
& + \left[ \left( -32 A(u)^2 z^2 w^2 - 8 (A(u))^2 z^3 w \frac{dA}{du} - 24 r z^{5/2} \sqrt{w} A(u) \right) Q(x^a)^2 \right. \\
& \left. + \left( -8 zw - 8 \frac{dA}{du} z^2 \right) Q(x^a) \right] L(x^a) - 8 \frac{dA}{du} zw Q(x^a) \\
& + \left( -12 r A(u) z^{3/2} w^{3/2} - 8 zw^3 A(u)^2 - 8 A(u)^2 z^2 w^2 \frac{dA}{du} A(u) \right) Q(x^a)^2
\end{aligned}$$

And  $(c')$  gives  $\Psi_3$  (as do several others):

```
> -thorn(kappa1)+conj(tau)*rho1+sigma1*tau:
> Psi3:=diffsbs(%);
```

$$\begin{aligned}\Psi_3 := & 6 Q(x^a)^2 L(x^a)^2 A(u)^2 z^{13/4} w^{1/4} \\ & + \left( \left( 14 (A(u))^2 z^{9/4} w^{5/4} + 6 \frac{A(u) z^{11/4} r}{w^{1/4}} \right) Q(x^a)^2 + 2 Q(x^a) z^{5/4} w^{1/4} \right) L(x^a) \\ & + \left( 2 A(u)^2 z^{9/4} w^{5/4} \frac{dA}{du} + 6 z^{5/4} w^{9/4} A(u)^2 + 6 A(u) z^{7/4} w^{3/4} r \right) Q(x^a)^2 \\ & + 2 \frac{dA}{du} z^{5/4} w^{1/4} Q(x^a)\end{aligned}$$

Now we check the leading terms (in inverse powers of  $r$ ) of our  $\Psi_3$  and  $\Psi_4$  and compare to [1].

```
> subs(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi4)):
> T4:=subs(r=1/R,%):
> series(T4,R=0,3) assuming R::positive;
```

$$-8 \frac{dA}{du} z w R^2 + O(R^3)$$

Here the leading term agrees with [1]. Next  $\Psi_3$

```
> subs(L(u,r,z,w)=Lis,Q(u,r,z,w)=Qis,Hlsbs(Psi3)):
> T3:=subs(r=1/R,%):
> series(T3,R=0,4) assuming R::positive;
```

$$2 \frac{dA}{du} z^{5/4} w^{1/4} R^2 + 8 z^{7/4} w^{3/4} A(u) R^3 + O(R^4)$$

We find that the leading term agrees with [1], but in the second term the powers of  $z = \zeta$  and  $w = \bar{\zeta}$  are wrong in [1]. We can also check that the Bianchi identities, [4] (4.12.36-39) are satisfied (and they are).

Finally, we turn to the remaining integrability conditions, [3] (10e) to (10h).

```
> P110sbs2(diffsbs(Psi2*Hmb-diff(p00,r)-Psi3*Hl(u))); # [3] 10e
0
> Psi3*Hm-Psi2*Hn-2*rho1*p01+2*beta*p00+del(p00): # [3] 10f
> P110sbs2(diffsbs(%));
0
> Psi3*Hmb-Psi4*Hl(u)+2*sigma1*p01-2*alpha*p00-del1(p00): # [3] 10g
> P110sbs2(diffsbs(%));
0
> Psi4*Hm-Psi3*Hn-2*kappa1*p01+2*gamma*p00+D1(p00): ## [3] 10h
```

```
> Hlsbs(P110sbs2(diffsbs(%)));
```

0

So we see that all the homothetic and Killing equations are satisfied and we have shown that there is always a Killing vector in these metrics and also always a homothety.

## 4 The Calculations (limit cylindrical case)

Since neither [1] not [2] give the spin coefficients for the limit cylindrical metric, we will need to calculate them using Maple's `tensor` package. Note that we use the corrected version of this metric, see [5] equation (26.23)

```
> with(tensor):
> coord:=[u,r,x,y]:g_c:=array(1..4,1..4,symmetric,sparse):
> g_c[1,1]:=-expand(simplify((b+log(r^2*x^4))/x^2/2)
> assuming r::positive,x::positive);
```

$$g_{-C_{1,1}} := -\frac{b}{2x^2} - \frac{\ln(r)}{x^2} - 2\frac{\ln(x)}{x^2}$$

```
> g_c[1,2]:=1:g_c[3,3]:=-2*r^2:g_c[4,4]:=-2*x^2:g_c[1,3]:=2*r/x:
> g:=create([-1,-1],eval(g_c)):
```

Next we calculate all the relevant tensors.

```
> tensorsGR(coord,g,gup,'detg','C1','C2','Rm','Rc','R','G','C');
```

To calculate the spin coefficients, I use a set of routines available on my web site <http://www.maths.unsw.edu.au/~jds/papers.html>

```
> read 'PRcoeff':
```

Now we define the tetrad, with the choice of  $m^a$  dictated by the need for the tetrad to be right-handed, so the anti-self duality used in the definition of the homothetic bivector (see [3]) is satisfied.

```
> md:=create([-1],vector([0,0,r,-I*x])):mup:=raise(gup,md,1):
> mbd:=create([-1],vector([0,0,r,I*x])):mbup:=raise(gup,mbd,1):
> ld:=create([-1],vector([1,0,0,0])):lup:=raise(gup,ld,1):
> nd:=create([-1],vector([g_c[1,1]/2,1,2*r/x,0])):nup:=raise(gup,nd,1):
> his:=linalg[stackmatrix](ld[compts],nd[compts],md[compts],mbd[compts]):
> h:=create([1,-1],op(his)):
```

Using the routines `PRspin` and `PRcrv` from the `PRcoeff` file we calculate the spin coefficients, and curvature components.

```
> spins:=PRspin(g,h,C2,coord):
> crv:=PRcurve(g,h,C,Rc,coord);
```



From these two calculations we find that the non-zero spin coefficients are

$$\tau = \beta = \tau' = -\frac{1}{2rx}, \quad \rho = \sigma = -\frac{1}{2r}, \quad \gamma = -\frac{1}{4rx^2}, \quad \rho' = \sigma' = \frac{b + \log(r^2 x^4)}{8rx^2};$$

and the non-zero curvature components are

$$\Psi_1 = \frac{1}{2r^2 x}, \quad \Psi_2 = \frac{1}{2r^2 x^2}, \quad \Psi_3 = \frac{b + \log(r^2 x^4)}{8r^2 x^3}.$$

Now we define the derivative operators  $D$ ,  $D'$ ,  $\delta$  and  $\delta'$

```
> De:=XX->add(lup[compts][i]*diff(XX,coord[i]),i=1..4):
> D1:=XX->add(nup[compts][i]*diff(XX,coord[i]),i=1..4):
> del:=XX->add(mup[compts][i]*diff(XX,coord[i]),i=1..4):
> del1:=XX->add(mbup[compts][i]*diff(XX,coord[i]),i=1..4):
```

Now to find the Killing vectors. Using [3] (6a),(6c) and (11) gives  $\xi_\ell = \xi_\ell(u)$ . Then from [3] (10a) we get  $\phi_{11}$ , and find that  $\phi_{11}^0(u, x, y)$ , the integration constant, is real by [3] (6g), which also gives  $\xi_m$ . So

```
> Hm:=-r*p110(u,x,y)+I*Hm0(u,x,y):Hmb:=-r*p110(u,x,y)-I*Hm0(u,x,y):
> p11:=Hl(u)/2/x/r+p110(u,x,y); # note that p110 is real
```

$$p11 := \frac{Hl(u)}{2xr} + p110(u, x, y)$$

We also solve [3] (10b) for  $\phi_{01}$ .

```
> crv[Psi1]*Hm-2*spins[sigma]*p01-(spins[beta]-spins[alpha1])*p11+del(p11):
> p01:=expand(solve(%,p01));
```

$$p01 := -\frac{p110(u, x, y)}{2x} - i\frac{Hm0(u, x, y)}{2xr} - \frac{Hl(u)}{4x^2 r} + \frac{1}{2}\frac{\partial}{\partial x}p110(u, x, y) - i\frac{r}{2x}\frac{\partial}{\partial y}p110(u, x, y)$$

Now looking at [3] (8a), using the fact that  $\kappa = 0$ :

```
> diff(p01,r)+del1(p11)-2*spins[rho]*p01-
> (spins[tau1]+spins[alpha]-spins[beta1])*p11:
> expand(%)
```

$$-\frac{3i}{2x}\frac{\partial}{\partial y}p110(u, x, y)$$

So  $\phi_{11}^0$  is independent of  $y$ . Now looking at [3] (6b):

```
> D1(Hl(u))+2*spins[epsilon1]*Hl(u)+spins[tau]*Hm+spins[tau]*Hmb+p01
> +subs(I=-I,p01)-psi:
> collect(%,r);
```

$$\frac{d}{du}Hl(u) + \frac{\partial}{\partial x}p110(u, x, y) - \psi$$

So we solve this for  $\phi_{11}^0$ , recalling that  $\phi_{11}^0$  is independent of  $y$ , and use it to redefine  $\phi_{11}$ ,  $\phi_{01}$  and  $\xi_m$ .

```
> p11:=Hl(u)/2/x/r+(psi-diff(Hl(u),u))*x+p0(u): # note that p0 is real
> p01:=expand(subs(p110(u,x,y)=(psi-diff(Hl(u),u))*x+p0(u),p01)):
> Hm:=expand(subs(p110(u,x,y)=(psi-diff(Hl(u),u))*x+p0(u),Hm));
```

$$Hm := -rx\psi + rx\frac{d}{du}Hl(u) - rp0(u) + iHm0(u, x, y)$$

```
> Hmb:=expand(subs(p110(u,x,y)=(psi-diff(Hl(u),u))*x+p0(u),Hmb)):
Turning to [3] (10c)
```

```
> crv[Psi2]*Hl(u)-crv[Psi1]*Hmb-2*spins[rho]*p01-(spins[alpha]-spins[beta1])*p11
> +del1(p11):
> expand(%);
```

0

Now the right hand side of [3] (6d) is

```
> -2*spins[epsilon]*Hn-spins[tau1]*Hm-spins[tau1]*Hmb+p01+subs(I=-I,p01)+psi:
> expand(%);
```

$$\frac{d}{du}Hl(u) - 2\frac{p0(u)}{x} - \frac{Hl(u)}{2x^2r}$$

This is  $D\xi_n$ , so we integrate

```
> int(%,r);
```

$$r\frac{d}{du}Hl(u) - 2\frac{r}{x}p0(u) - \frac{Hl(u)\ln(r)}{2x^2}$$

```
> Hn:=%+Hn0(u,x,y):
```

Turning to [3] (6i),

```
> del(Hm)+spins[sigma1]*Hl(u)+spins[sigma]*Hn+(spins[alpha1]+spins[alpha])*Hm:
```

The coefficients of  $r$  are independent, so we collect the terms.

```
> collect(expand(%),r);
```

$$\begin{aligned} & -\psi + \frac{1}{2}\frac{d}{du}Hl(u) - \frac{p0(u)}{2x} - \frac{1}{2x}\frac{\partial}{\partial y}Hm0(u, x, y) \\ & - \left( \frac{1}{2}i\frac{\partial}{\partial x}Hm0(u, x, y) + \frac{Hl(u)b}{8x^2} + \frac{Hl(u)\ln(x)}{2x^2} - \frac{iHm0(u, x, y)}{2x} + \frac{1}{2}Hn0(u, x, y) \right) r^{-1} \end{aligned}$$

The imaginary part of the  $r^{-1}$  term implies  $\xi_m^0 = xf(u, y)$ , for some function  $f(u, y)$ ,  
so:

> X:=collect(subs(Hm0(u,x,y)=x\*f(u,y),%),r):  
> Y:=solve(op(5,X),Hn0(u,x,y));

$$Y := -Hl(u) \frac{b + 4 \ln(x)}{4x^2}$$

> expand(subs(Hn0(u,x,y)=Y,X));

$$-\psi + \frac{1}{2} \frac{d}{du} Hl(u) - \frac{p0(u)}{2x} - \frac{1}{2} \frac{\partial}{\partial y} f(u, y)$$

> XX:=rhs(dsolve(%,f(u,y)))\*x;

$$XX := \left( -2y\psi + y \frac{d}{du} Hl(u) - \frac{yp0(u)}{x} + F1(u) \right) x$$

Where F1 is an arbitrary function. This XX is  $\xi_m^0$ . So

> Hm:=collect(subs(Hm0(u,x,y)=XX,Hm),[r,x,y]);  
> Hmb:=collect(subs(Hm0(u,x,y)=XX,Hmb),[r,x,y]):

$$Hm := \left[ \left( -\psi + \frac{d}{du} Hl(u) \right) x - p0(u) \right] r + \left[ i \left( -2\psi + \frac{d}{du} Hl(u) \right) y + i F1(u) \right] x - iy p0(u)$$

> Hn:=subs(Hn0(u,x,y)=Y,Hn);

$$Hn := \left( \frac{d}{du} Hl(u) \right) r - 2 \frac{p0(u)r}{x} - \frac{Hl(u)}{4x^2} [2 \ln(r) + b + 4 \ln(x)]$$

> p01:=expand(subs(Hm0(u,x,y)=XX,p01));

$$p01 := \frac{p0(u)}{2x} + \frac{iy\psi}{r} - \frac{iy}{2r} \frac{d}{du} Hl(u) + \frac{iy}{2rx} p0(u) - \frac{i}{2r} F1(u) - \frac{Hl(u)}{4x^2 r}$$

Returning to the integrability conditions, we look at [3] (10d)

> eq10d:=crv[Psi2]\*Hm-crv[Psi1]\*Hn-2\*spins[tau]\*p01-2\*spins[gamma]\*p11+D1(p11):  
> expand(%);

$$\frac{p0(u)}{2x^2 r} + \frac{d}{du} p0(u) - x \frac{d^2}{du^2} Hl(u)$$

So by comparing coefficients we have

> p0(u):=0;Hl:=x->k0\*x+k1;

$$p0(u) := 0$$

$$Hl := x \mapsto k0x + k1$$

And a quick check shows that eqn10d ([3] (10d)) is satisfied. Next, the conjugate of [3] (6h) will give us  $\phi_{00}$ .

```
> D1(Hmb)+spins[tau]*Hn+(spins[gamma]+spins[epsilon1])*Hmb:
> p00:=collect(expand(%),[psi,k1,k0]);
```

$$p00 := \left( -\frac{b}{4x} - \frac{\ln(r)}{2x} - \frac{\ln(x)}{x} \right) \psi + \left( \frac{\ln(r)}{4rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3} \right) k1 +$$

$$\left( \frac{ub}{8rx^3} + \frac{b}{4x} - \frac{1}{2x} + \frac{\ln(r)}{2x} + \frac{\ln(x)}{x} + \frac{u \ln(x)}{2rx^3} + \frac{\ln(r)u}{4rx^3} \right) k0 - i \left( \frac{d}{du} F1(u) \right) x$$

We next check some further integrability conditions, [3] (10e) first.

```
> crv[Psi3]*H1(u)-crv[Psi2]*Hmb-2*spins[tau1]*p01+2*spins[epsilon]*p00
> +diff(p00,r):
> expand(%);
```

0

And then [3] (8d).

```
> e8d:=diff(p00,r)+del1(p01)-spins[rho]*p00-2*spins[tau1]*p01-spins[sigma1]*p11:
> expand(%);
```

$$-\frac{ix}{2r} \left( \frac{d}{du} F1(u) \right)$$

So the integrability function  $F1$  is constant:

```
> _F1(u):=k3;expand(e8d);
```

$$\_F1(u) := k3$$

0

Also [3] (6e) is

```
> eqn6e:=expand(D1(Hn)+2*spins[gamma]*Hn);
```

$$eqn6e := -\frac{k0}{2x^2}$$

Thus  $k0 = 0$ , and the coefficients simplify as follows:

```
> H1(u);Hn;Hm;
```

$$\frac{k1}{2x^2} - \frac{k1(b + 4 \ln(x))}{4x^2}$$

$$-rx\psi + (-2iy\psi + ik3)x$$

> p00;p01;p11;

$$\left(-\frac{b}{4x} - \frac{\ln(r)}{2x} - \frac{\ln(x)}{x}\right)\psi + \left(\frac{\ln(r)}{rx^3} + \frac{\ln(x)}{2rx^3} + \frac{b}{8rx^3}\right)k1$$

$$\frac{iy\psi}{r} - \frac{ik3}{2r} - \frac{k1}{4x^2r}$$

$$\frac{k1}{2rx} + x\psi$$

All the remaining homothetic equations and integrability equations are satisfied, and we are left with the general homothetic vector:

> lin\_com(H1(u), nup, Hn, lup, -Hm, mbup, -Hmb, mup);

TABLE ([index\_char = [1], compts = vector ([k1, 2rψ, -xψ, 2yψ - k3]))

That is,

$$k_1\partial_u + k_3\partial_y + \psi(2r\partial_r - x\partial_x + 2y\partial_y).$$

## 5 Acknowledgments

Maple is a registered trademark of Waterloo Maple Inc.

## 6 References

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